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N-TYPES OF SIMPLICIAL GROUPS AND CROSSED N-CUBES

TIMOTHY PORTER

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AS IS well known, simplicial groups model all connected homotopy types. In particular certain simplicial groups, namely those with vanishing Moore complex in dimensions greater than n, provide algebraic models for n-types of simplicial groups and thus for connected $(n+1)$-types of spaces. In [12], Loday gave the foundation of a theory of another algebraic model for $(n+1)$-types that generalized that given for 2-types by MacLane and Whitehead [13]. His models, called cat^n-groups, have very pleasant properties and in work with Brown [3], [4] have been shown to satisfy a form of generalized Van Kampen theorem. These cat^n-groups form a category equivalent to that of the crossed n-cubes of the title of this paper (c.f. Ellis and Steiner [9]). Loday's original result was stated using a theory of n-cubes of fibrations. This made it difficult to generalize to other algebraic contexts which clearly may be useful for further development in both homotopy theory and homotopical algebra. The aim of this paper is thus to provide full, detailed, purely algebraic proofs of Loday's main results and in the process prove stronger results that should prove of independent interest.

The key result, missed by Loday, although he knew all the ingredients to prove it, is that any crossed module can be obtained as π_0 of a normal inclusion, $N \to G$, of simplicial groups. This generalizes easily to crossed n-cubes: each crossed n-cube is isomorphic to π_0 of a simplicial inclusion crossed n-cube determined by a simplicial group $G.$ and n-normal subgroups $N(1).,\ldots, N(n)$. Thus any crossed square can be obtained by taking π_0 of a square of the form:

$$\begin{array}{ccc} N(1)\cap N(2). & \to & N(1). \\ \downarrow & & \downarrow \\ N(2). & \to & G. \end{array}$$

The category of simplicial inclusion crossed n-cubes is linked to that of simplicial groups by a pair of functors Q: Simp(Inc.Crs^n) $\to$ Simp.Gps. and $\mathcal{M}(-,n)$ in the other sense. For $n=2$, Q of the above simplicial inclusion crossed square is $G./N(1).N(2).$. One of the main results of this paper (Theorem 2) states:

The functors $\mathcal{M}$ and Q set up an equivalence between the homotopy category of simplicial groups and a quotient category of Simp(*Inc.Crs^n*).

The quotient is by a class of quasi-isomorphisms (see Section 6 for the definition).

Loday's result in [12], is somewhat similar in form, although the construction he attempts to use does not quite work. Applying π_0 to $\mathcal{M}(-,n)$ gives a functor $M(-,n)$ from Simp.Gps. to Crs^n. A quasi-inverse for this is given by a "multinerve" functor E. The resulting simplicial group is an n-type, i.e. has vanishing π_i for $i > n$ and there are zig-zag chains of mappings joining a $G.$ to $EM(G., n)$. The other main result of this paper, Theorem 1, states that E and $M(-,n)$ induce an equivalence between Ho_n(Simp.Gps), the homotopy category of n-types of simplicial groups, and a category formed from Crs^n by inverting the quasi-isomorphisms.

In a future paper, it will be shown how to develop models that combine features both of these crossed n-cubes and the crossed complexes of Brown and Higgins [2].

1. PRELIMINARIES ON SIMPLICIAL GROUPS

Given a simplicial group $G.$, the Moore Complex (NG, ∂) of $G.$ is the chain complex defined by

$$(NG)_n = \cap(\mathrm{Ker} d_i^n \colon i \neq 0)$$

with $\partial_n \colon NG_n \to NG_{n-1}$ induced from d_0^n by restriction. The image $d_0(NG_{n+1})$ is normal in G_n and

$$\pi_n(G.) \cong H_n(NG, \partial)$$

(see results on the Moore Complex in most standard books on simplicial homotopy theory or in the survey article of Curtis [7].)

The fact that each composite $d_{i+1}\, s_i$ is the identity allows one to decompose any G_n into a semi-direct product of degenerate images of lower order Moore Complex terms and NG_n itself. More precisely, one has

Proposition 1 (Conduché [5]). *If $G.$ is a simplicial group, then for any $n \geq 0$, G_n is isomorphic to* $NG_n \rtimes s_0 NG_{n-1} \rtimes s_1 NG_{n-1} \rtimes s_1 s_0 NG_{n-2} \rtimes s_2 NG_{n-1} \rtimes \ldots \rtimes s_{n-1} s_{n-2} \ldots s_0 NG_0$.

The order of terms corresponds to a lexicographic ordering of the indices: $\varnothing$; 0; 1; 1, 0; 2; 2, 0; 2, 1; 2, 1, 0; 3; 3, 0; ... and so on. The term corresponding to $i_1 > \ldots > i_p$ is $s_{i_1} \ldots s_{i_p}(NG_{n-p})$. The bracketting of the terms will be evident from the construction given below.

Proof.
First since $d_n\, s_{n-1} = id$,

$$G_n \cong \mathrm{Ker}\, d_n \rtimes s_{n-1}(G_{n-1})$$

For inductive hypothesis; assume the result for $n-1$ and all simplicial groups. Note that $\mathrm{Ker}\, d_{\mathrm{last}}$ is a simplicial group using the other faces and $(\mathrm{Ker}\, d_{\mathrm{last}})_{n-1} = \mathrm{Ker}\, d_n$ so we have the decomposition for both the above factors. The details here are easy to check. Finally we note that the case $n = 0$ is trivial, completing the proof of the result.

If $G.$ is abelian, then this result is part of the proof of the Dold–Kan theorem. (c.f. Curtis [7] or Lamotke [11]).

Given any simplicial group $G.$, $\mathrm{Dec}'(G)$, is the augmented simplicial group obtained from $G.$ by forgetting the last face and degeneracy operators at each level and then renumbering the levels (c.f. Duskin [8] or Illusie [10]). Thus $(\mathrm{Dec}\, G)_n = G_{n+1}$. The last degeneracy of $G.$ yields a contraction of $\mathrm{Dec}^1 G.$ as an augmented simplicial group, $\mathrm{Dec}^1 G. \cong K(G_0, 0)$, by an explicit natural homotopy equivalence (c.f. Duskin [8]). The last face map will be denoted $\delta_0 \colon \mathrm{Dec}^1 G. \to G.$ and has kernel the simplicial group, $\mathrm{Ker}\, d_{\mathrm{last}}$, used above.

Iterating the Dec construction yields as augmented bisimplicial group

$$\left(\cdots \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{Dec}^3 G. \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{Dec}^2 G. \substack{\longrightarrow \\ \longrightarrow} \mathrm{Dec}\, G. \right)$$

which in expanded form is the total décalage of $G.$:

$$
\begin{array}{ccccccc|ccc}
G_6 & \Rrightarrow & G_5 & \Rrightarrow & G_4 & & & \xrightarrow{d_4} & & G_3 \\
\downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & & & & \downarrow\downarrow\downarrow\downarrow \\
G_5 & \Rrightarrow & G_4 & \Rrightarrow & G_3 & & & \xrightarrow{d_3} & & G_2 \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & & & & \downarrow\downarrow\downarrow \\
G_4 & \Rrightarrow & G_3 & \Rrightarrow & G_2 & & & \xrightarrow{d_2} & & G_1 \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & & & & \downarrow\downarrow \\
G_3 & \Rrightarrow & G_2 & \Rrightarrow & G_1 & & & \xrightarrow{d_1} & & G_0
\end{array}
$$

(see Duskin [8] or Illusie [10] for details). The maps from $\mathrm{Dec}^i G.$ to $\mathrm{Dec}^{i-1} G.$ coming from the ith last face maps will be labelled $\delta., \ldots, \delta_{i-1}$ so that $\delta_0 = d_{\text{last}}$, $\delta_1 = d_{\text{last but one}}$ and so on.

By a normal chain complex of groups, (X, d) we mean one in which each $\mathrm{Im}\, d_{i+1} \lhd X_i$. Given any normal chain complex (X, d) of groups and an integer n the truncation, $t_{n]}X$, of X at level n will be defined by

$$
(t_{n]}X)_i = \begin{cases} X_i & \text{if } i < n \\ X_i/\mathrm{Im}\, d_{n+1} & \text{if } i = n \\ 0 & \text{if } i > n \end{cases}
$$

The differential d of $t_{n]}X$ is that of X for $i < n$, d_n is induced from the nth differential of X and all others are zero.

PROPOSITION 2. *There is a truncation functor $t_{n]}$: Simp.Gps. $\to$ Simp.Gps. such that there is a natural isomorphism*

$$t_{n]}N \cong N\, t_{n]}$$

where N is the Moore complex functor.

Proof. We take:

$$(t_{n]}G)_i = G_i \quad \text{if } i < n$$

$$(t_{n]}G)_n = G_n/d_0(NG_{n+1})$$

and if $i > n$, we use the decomposition of G_n, delete all NG_k for $k > n$ and replace NG_n by $NG_n/d_0(NG_{n+1})$. The face and degeneracy maps are now easy to define. The remaining details are omitted as they are routine.

We note the following properties of $t_{n]}$.

PROPOSITION 3. *Let $T_{n]}$ be the full subcategory of Simp.Gps. defined by those simplicial groups whose Moore Complex is trivial in dimensions greater than n. Let i_n: $T_{n]} \to$ Simp.Gps be the inclusion functor, then*

(a) *$t_{n]}$ is left adjoint to i_n;*

(b) *the counit of this adjunction is a natural epimorphism which induces an isomorphism on π_i for $i \leq n$;*

(c) *for any simplicial group $G.$, $\pi_i(t_{n]}G.) = 0$ if $i > n$;*

The inclusion $T_{n]} \to T_{n+1]}$ corresponds to a natural epimorphism $\eta_n\, t_{n+1]}$ to $t_{n]}$ and if $G.$ is a simplicial group, then Ker $\eta_n(G.)$ is a $K(\pi_{n+1}(G.), n+1)$.

Each of these statements is a simple consequence of the definitions.

Given the similarity if these properties with those of the coskeleton functors (as studied by Artin–Mazur [1]), it is important to note the following:

PROPOSITION 4. *Let $G.$ be a simplicial group, and $cosk_{n+1}\, G.$, the group-theoretic $(n+1)$-coskeleton of $G.$ (i.e. calculated within Simp.Gps). Then there is a natural epimorphism from $cosk_{n+1}\, G.$ to $t_{n]}\, G.$ with acyclic kernel. Thus $cosk_{n+1}\, G.$ and $t_{n]}G.$ have the same weak homotopy type.*

Proof. Following Conduché [5], the Moore complex of $\mathrm{cosk}_{n+1}\, G.$ is given by:

$$N(\mathrm{cosk}_{n+1}\, G)_r = 0 \quad \text{of } r > n+2,$$
$$N(\mathrm{cosk}_{n+1}\, G)_{n+2} = \mathrm{Ker}(\partial_{n+1}\colon NG_{n+1} \to NG_n),$$
$$N(\mathrm{cosk}_{n+1}\, G)_r = NG_r \quad \text{if } r \leq k+1.$$

The natural epimorphism gives on Moore complexes

$$\begin{array}{llllll}
N(\cos k_{n+1} G)\colon & 0 \longrightarrow \mathrm{Ker}\,\partial_{n+1} & \longrightarrow NG_{n+1} & \xrightarrow{\partial_{n+1}} NG_n & \longrightarrow NG_{n-1} \\
\downarrow & \qquad\quad \downarrow & \qquad \downarrow & \qquad \downarrow & \qquad \downarrow \\
N(t_{n]}G)\colon & 0 \longrightarrow 0 & \longrightarrow 0 & \longrightarrow NG_n/\mathrm{Im}\,\partial_{n+1} & \longrightarrow NG_{n-1}
\end{array}$$

and it is immediate that the kernel is exact as required.

For the purpose of this paper $t_{n]}$ is more convenient than cosk_{n+1}. A morphism $f.\colon G. \to H.$ of simplicial groups will be called an n-equivalence if $\pi_i(f)\colon \pi_i(G.) \to \pi_i(H.)$ is an isomorphism for all i, $0 \leq i \leq n$. Two groups $G.$ and $H.$ are said to have the same n-type if there is a chain of n-equivalences linking them. If Ho_n(Simp.Gps.) is obtained by formally inverting the n-equivalences, then $G.$ and $H.$ have the same n-type if and only if they are isomorphic in Ho_n(Simp.Gps.). Any simplicial group $G.$ has the same n-type as $t_{n]}G$ (by Proposition 3) (and as $\mathrm{cosk}_{n+1}G$ by Proposition 4). Truncated simplicial groups thus provide models for n-types. In the next section crossed n-cubes are introduced which provide an alternative model.

2. CROSSED n-CUBES

The following definition is due to Ellis and Steiner [9]. Let $\langle n\rangle$ denote the set $\{1, \ldots, n\}$.

A crossed n-cube of groups is a family $\{M_A\colon A \subseteq \langle n\rangle\}$ of groups, together with homomorphisms $\mu_i\colon M_A \to M_{A-\{i\}}$ for $i \in \langle n\rangle$, $A \subseteq \langle n\rangle$ and functions $h\colon M_A \times M_B \to M_{A\cup B}$ for $A, B \subseteq \langle n\rangle$, such that if ${}^a b$ denotes $h(a, b,)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \langle n\rangle$, the following axioms hold:

(1) $\mu_i a = a$ if $i \notin A$

(2) $\mu_i \mu_j a = \mu_j \mu_i a$

(3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$

(4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$

(5) $h(a, a') = [a, a']$

(6) $h(a, b) = h(b, a)^{-1}$

(7) $h(a, b) = 1$ if $a = 1$ or $b = 1$

(8) $h(aa', b) = {}^a h(a', b)\, h(a, b)$

(9) $h(a, bb') = h(a, b)\, {}^b h(a, b')$

(10) ${}^a h(h(a^{-1}, b), c)\, {}^c h(h(c^{-1}, a), b)\, {}^b h(h(b^{-1}, c), a) = 1$

(11) ${}^a h(b, c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$

A morphism of crossed n-cubes

$$\{M_A\} \to \{M'_A\}$$

is a family of morphisms $\{f_A\colon M_A \to M'_A, A \subseteq \langle n \rangle\}$ of groups which commute with the μ_i and the h-functions. This gives a category which will be denoted Crs^n. Ellis and Steiner [9] prove that Crs^n is equivalent to the category of cat^n-groups introduced by Loday in [12]. The h-map identities are modelled on commutator identities.

Example. A crossed 1-cube is the same as a crossed module. The detailed reformulation is easy.

PROPOSITION 5. *Let $\mathcal{M} = \{M_A\colon A \subseteq \langle n \rangle, \{\mu_i\}, h\}$ be a crossed n-cube of groups. For any $i \in \langle n \rangle$, let $\mathcal{M}_1$ denote the family $\{M_A\colon A \subseteq \langle n \rangle, i \in A\}$ and $\mathcal{M}_0$ the corresponding family with the condition $i \notin A$. Then*

(a) *$\mathcal{M}_0$ and $\mathcal{M}_1$ both have the structure of crossed $(n-1)$-cubes of groups.*

(b) *$\mu_i\colon \mathcal{M}_1 \to \mathcal{M}_0$ is a morphism of crossed $(n-1)$-cubes such that for each $B \subseteq \langle n-1 \rangle$, $\operatorname{Ker} \mu_{i,B}$ is central in $\mathcal{M}_{1,B}$ and $\operatorname{Im} \mu_{i,B}$, $\subseteq \mathcal{M}_{0,B}$, is normal.*

(c) *denoting $\mathcal{N}_B = \mathcal{M}_{0,B}/\operatorname{Im} \mu_{i,B}$, $B \leq \langle n-1 \rangle$, then $\mathcal{N} = \{N_B\colon B \subseteq \langle n-1 \rangle\}$ has the natural structure of a crossed $(n-1)$-cube and the sequence*

$$0 \to \operatorname{Ker} \mu_i \to \mathcal{M}_1 \xrightarrow{\mu_i} \mathcal{M}_0 \to \mathcal{N} \to 1$$

is an exact sequence of crossed $(n-1)$-cubes in the obvious sense.

Proof. (a) is mostly routine requiring only care over the notation. For (b) the first part is routine, the centrality and normality statements are the analogues of the well known fact that if $\mu\colon C \to P$ is a crossed module $\operatorname{Ker}\mu$ is central in C and $\operatorname{Im} \mu$ is normal in P. They are proved in the same way. (c) is routine checking of well definition of induced maps and functions, followed by verification of the axioms.

Example and definition. Let G be a group and $N_1, \ldots, N_n$ normal subgroups of G. For $A \subseteq \langle n \rangle$, let $M_A = \underset{i \in A}{\cap} N_i$; if $i \in \langle n \rangle$, define $\mu_i\colon M_A \longrightarrow M_{A-\{i\}}$ to be the inclusion and given $A, B, \subseteq \langle n \rangle$, let $h\colon M_A \times M_B \to M_{A \cup B}$ be given by the communicator: $h(a, b) = [a, b]$.

Then $\{M_A\colon A \subseteq \langle n \rangle, \mu_i, h\}$ is a crossed n-cube, called the inclusion crossed n-cube given by the normal $(n+1)$-ad of groups $(G; N_1, \ldots, N_n)$. The category of inclusion crossed n-cubes will be denoted, $\mathrm{Inc.Crs}^n$.

By way of illustrating Proposition 5, note that if $(G; N_1, N_2)$ is given, the corresponding inclusion crossed square is

$$\begin{array}{ccc} N_1 \cap N_2 & \to & N_2 \\ \downarrow & & \downarrow \\ N_1 & \to & G \end{array}$$

and the quotient (horizontally) is isomorphic to the inclusion crossed module.

$$\frac{N_2 N_1}{N_1} \to \frac{G}{N_1}.$$

The quotienting construction illustrated for the crossed square above can be repeated in each of the n-direction of a crossed n-cube eventually giving a group. This gives a functor

$$Q\colon \mathrm{Crs}^n \to \mathrm{Gps}.$$

For instance, in the above example $Q(\mathscr{M}) \cong G/N_1N_2$ by the isomorphism theorems of group theory. Similarly if $\mathscr{M}$ is constructed from $(G;\ N_1, \ldots, N_n)$ then $Q(\mathscr{M}) \cong G/N_1 \ldots N_n$. However it is useful to be able to break up this into n iterations of the simple quotienting operation. The functor Q extended to Simp(Inc.Crs^n) and taking values in Simp.Gps. will be needed later on.

Proposition 6. *Let $(G.;\ N_1., \ldots, N_n.)$ be a simplicial normal $(n+1)$-ad of groups and define for $A \subseteq \langle n \rangle$*

$$M_A = \pi_0(\cap\{N_i.\colon i \in A\})$$

with homomorphisms $\mu_i\colon M_A \to M_{A-\{i\}}$ and h-maps induced by the corresponding maps in the simplicial inclusion crossed n-cube, constructed by applying the previous construction to each level. Then $\mathscr{M} = \{M_A\}$ is a crossed n-cube.

Proof. The functor π_0 does not destroy the structures involved as they are given using finite products. (An alternative proof can be constructed using routine calculations of well definition and verification of axioms).

Later on it will be proved that, up to isomorphism, all crossed n-cubes arise in this way.

3. FROM SIMPLICIAL GROUPS TO CROSSED n-CUBES

Proposition 6 above gives the key to defining for each n, a functor from simplicial groups to crossed n-cubes that embeds $T_{n]}$ into Crs^n. The idea is adapted from one of Loday and the author also benefited from talking to Bullejos and Duskin. Another approach to this construction is given in Cordier and Porter [6].

For each simplicial group, $G.$, there is a functorial short exact sequence

$$\mathrm{Ker}\ \delta_0 \longrightarrow \mathrm{Dec}^1\ G \longrightarrow G$$

corresponding to the 0-skeleton of the total décalage of G.

The 1-skeleton of that total décalage gives, a 3×3 diagram with exact rows and columns:

$$\begin{tikzcd}
\mathrm{Ker}\ \delta_0 \cap \mathrm{Ker}\ \delta_1 \arrow[r] \arrow[d] & \mathrm{Ker}\ \delta_1 \arrow[r] \arrow[d] & \mathrm{Ker}\ \delta_0 \arrow[d] \\
\mathrm{Ker}\ \delta_0 \arrow[r] \arrow[d] & \mathrm{Dec}^2\ G. \arrow[r, "\delta_0"] \arrow[d, "\delta_1"] & \mathrm{Dec}^1\ G. \arrow[d, "\delta_0"] \\
\mathrm{Ker}\ \delta_0 \arrow[r] & \mathrm{Dec}^1\ G. \arrow[r, "\delta_0"] & G.
\end{tikzcd}$$

and thus a simplicial inclusion crossed n-cube continuing this n-times given the simplicial inclusion crossed n-cube corresponding to the simplicial $(n+1)$-ad $(\mathrm{Dec}^n G;\ \mathrm{Ker}\ \delta_{n+1}, \ldots, \mathrm{Ker}\ \delta_0)$. This simplicial inclusion n-cube will be denoted $\mathscr{M}(G., n)$, whilst its associated crossed n-cube $\pi_0(\mathscr{M}(G., n))$ will be denoted $M(G., n)$. The construction of $M(G., n)$ gives the following:

Proposition 7. *Given a simplicial group, $G.$, the crossed n-cube $M(G., n)$ is given by:*

(a) *for $A \subseteq \langle n \rangle$,*

$$M(G., n)_A = \frac{\bigcap\{\mathrm{Ker}\ d_j^n : j \in A\}}{d_0(\mathrm{Ker} d_1^{n+1} \cap \bigcap\{\mathrm{Ker} d_{j+1}^{n+1} : j \in A\})};$$

(b) *if* $i \in \langle n \rangle$, *the homomorphism* $\mu_i\colon M(G., n)_A \to \mathcal{M}(G., n)_{A-\{i\}}$ *is induced from the inclusion of* $\bigcap\{\mathrm{Ker} d_j^n\colon j \in A\}$ *into* $\bigcap\{\mathrm{Ker} d_j^n\colon j \in A - \{i\}\}$;

(c) *representing an element in* $M(G., n)_A$ *by* $\bar{x}$ *where* $\bar{x} \in \bigcap\{\mathrm{Ker} d_j^n\colon j \in A\}$ *(so the bar denotes a coset), and for* $A, B \subseteq \langle n \rangle$, $\bar{x} \in M(G., n)_A$, $\bar{y} \in M(G., n)_B$, $h(\bar{x}, \bar{y}) = [\bar{x}, \bar{y}] \in M(G., n)_{A \cup B}$.

Proof. This follows from the description of π_0 as H_0 of the Moore complex. (The reverse order on the $(n+1)$-ad defining $\mathcal{M}(G., n)$ is there to ensure the formula in (a) is relatively simple).

PROPOSITION 8. *Let G. be a simplicial group. Then*

(a) *for* $A \subseteq \langle n \rangle, A \neq \langle n \rangle$,

$$M(G., n)_A \cong \bigcap\{\mathrm{Ker} d_{i-1}^{n-1}\colon i \in A\}$$

so that in particular $M(G., n)_\varnothing \cong G_{n-1}$; *in each case the isomorphism is induced by* d_0. *(It will be often convenient to identify these isomorphic groups).*

(b) *If* $A \neq \langle n \rangle$ *and* $i \in \langle n \rangle$, $\mu_i\colon M(G., n)_A \to M(G., n)_{A-\{i\}}$ *is the inclusion of a normal subgroup.*

(c) *for* $j \in \langle n \rangle$, $\mu_j\colon M(G., n)_{\langle n \rangle} \to \bigcap\{\mathrm{Ker} d_{i+1}^{n+1}\colon i \neq j\}$ *is induced by* d_0.

Proof. The following Lemma will be needed.

LEMMA A. *If G. is a simplicial group and* $A \subseteq \langle n \rangle, A \neq \langle n \rangle$, *then* $d_0(\bigcap\{\mathrm{Ker}\, d_i^n\colon i \in A\}) = \bigcap\{\mathrm{Ker} d_{i-1}^{n-1}\colon i \in A\}$.

Proof. Since $d_{i-1} d_0 = d_0 d_i$ if $i \in A$,

$$d_0(\bigcap\{\mathrm{Ker}\, d_i^n\colon i \in A\}) \subseteq \bigcap\{\mathrm{Ker}\, d_{i-1}^{n-1}\colon i \in A\}.$$

To prove the opposite inclusion, let K be the first integer in $\langle n \rangle \backslash A$ and suppose $x \in \bigcap\{\mathrm{Ker}\, d_{i-1}^{n+1}\colon i \in A\}$. Now let

$$y = (s_0 x)(s_1 x)^{-1} \ldots (s_{k-1} x)^{\varepsilon(k-1)}$$

where $\varepsilon(k-1) = (-1)^{k-1}$, then $d_0 y = x$ and $d_i y = 1$ for all $i \in A$.

Returning to the main proof, this lemma gives for $A \neq \langle n \rangle$,

$$M(G., n)_A = \frac{\bigcap\{\mathrm{Ker}\, d_i^n\colon i \in A\}}{\mathrm{Ker} d_0^n \cap \bigcap\{\mathrm{Ker} d_i^n\colon i \in A\}}$$

The epimorphism $d_0\colon G_n \to G_{n-1}$, which is split by s_0, restricts to give an epimorphism from $\bigcap\{\mathrm{Ker} d_i^n\colon i \in A\}$ onto $\bigcap\{\mathrm{Ker} d_{i-1}^{n-1}\colon i \in A\}$, again by the lemma. The kernel of this epimorphism is the denominator, which completes the proof of (a). Parts (b) and (c) are now simple consequences.

Combining Propositions 5 and 8 gives a connection between $M(G., n)$ and $M(G., n-1)$. The notation of Proposition 5 will be adopted for the case $i = 1$, so that $M(G., n)_1$ will denote the crossed $(n-1)$-cube obtained by restricting to those $A \subseteq \langle n \rangle$ with $1 \in A$, and $M(G., n)_0$ that obtained from the terms with $A \subseteq \langle n \rangle, 1 \notin A$.

PROPOSITION 9. *Given a simplicial group G. and $n \geq 1$, there is an exact sequence of crossed $(n-1)$-cubes:*

$$1 \to K. \to M(G., n)_1 \xrightarrow{\mu_1} M(G., n)_0 \to M(G., n-1) \to 1$$

where if $B \subseteq \langle n-1 \rangle$ and $B \neq \langle n-1 \rangle$, then $K_B = \{1\}$, whilst $K_{\langle n-1 \rangle} \cong \pi_n(G.)$

Proof. If $B \subseteq \langle n-1 \rangle$, set

$$B(1) = \{1\} \cup \{b+1 : b \in B\} \subseteq \langle n \rangle$$

and

$$B(0) = \{b+1 : b \in B\} \subseteq \langle n \rangle.$$

If $B \neq \langle n-1 \rangle$, then $B(1) \neq \langle n \rangle$ and Proposition 8 gives

$$\begin{aligned} M(G., n)_{1,B} = M(G., n)_{B(1)} &= \bigcap \{\operatorname{Ker} d_{i-1}^{n-1} : i \in B(1)\} \\ &= \operatorname{Ker} d_0^{n-1} \cap \bigcap \{\operatorname{Ker} d_{i-1}^{n-1} : i \in B(0)\} \\ &= \operatorname{Ker} d_0^{n-1} \cap M(G., n)_{0,B} \end{aligned}$$

with μ_1 being the inclusion. Thus for such a B, $K_B = (\operatorname{Ker} \mu_n)_B = \{1\}$ whilst $(\operatorname{Coker} \mu_n)_B$ is given by

$$\frac{\bigcap \{\operatorname{Ker} d_{i-1}^{n-1} : i \in B\}}{\operatorname{Ker} d_0^{n-1} \cap \bigcap \{\operatorname{Ker} d_{i-1}^{n-1} : i \in B\}} \cong M(G., n-1)_B$$

again by Proposition 8.

If $B = \langle n-1 \rangle$, then $B(1) = \langle n \rangle$,

$$M(G., n)_{1,B} = \frac{NG_n}{d_0(NG_{n-1})}$$

whilst

$$\begin{aligned} M(BG., n)_{0,B} &= \bigcap \{\operatorname{Ker} d_{i-1}^{n-1} : i \in B(0)\} \\ &= \bigcap \{\operatorname{Ker} d_i^{n-1} : i \in \langle n-1 \rangle\} \\ &= NG_{n-1}. \end{aligned}$$

as with these identifications μ_1 is induced by d_0, this gives

$$K_{\langle n-1 \rangle} = \frac{\operatorname{Ker} d_0 \cap NG_n}{d_0(NG_{n+1})} \cong H_n(NG) \cong \pi_n(G.)$$

Whilst $(\operatorname{Coker} \mu_1)_{\langle n-1 \rangle} = \dfrac{NG_{n-1}}{d_0(NG_n)} = M(G., n-1)_{\langle n-1 \rangle}$, as required.

The importance of this result is that it shows that $M(G., n)$ contains all the information to recover $\pi_i(G)$ for $i = 0, 1, \ldots, n$. In particular repeated use of this proposition proves that $Q(M(G., n)) \cong \pi_0(G.)$.

These results on $M(G., n)$ parallel results on $\mathscr{M}(G., n)$. The simplicial inclusion crossed n-cube $\mathscr{M}(G., n)$ was constructed using $\operatorname{Dec}^n G$ and the n-kernels $\operatorname{Ker} \delta_i$, $i = 0, 1 \ldots, n-1$. As all the μ_i's are inclusions, there is no immediately comparable term to $K.$ in the above, but it is clear that the cokernel of μ_1 is isomorphic to $\mathscr{M}(G., n-1)$.

PROPOSITION 10. *If G is any simplicial group and $n \geq 1$, then*

(a) *if $A \subseteq \langle n \rangle$, but $A \neq \langle n \rangle$, then $\mathscr{M}(G., n)_A$ is a $K(M(G, n)_A, 0)$ i.e. its only non-trivial homotopy group is π_0, which is $M(G., n)_A$.*

(b) *the simplicial group* $\mathscr{M}(G., n)_{\langle n \rangle} = H.$, *say, satisfies*

$$\pi_t(H.) \cong \pi_{n+t}(G.) \quad \text{for } t \geq 1$$

(c) *writing* $\mathscr{M}(G., n)_1$ *and* $\mathscr{M}(G., n)_0$ *for the two simplicial* $(n-1)$ *cubes in the* μ_1 *direction, the sequence*

$$1 \to \mathscr{M}(G., n)_1 \xrightarrow{\mu_1} \mathscr{M}(G., n)_0 \to \mathscr{M}(G., n-1) \to 1$$

is an exact sequence of simplicial inclusion $(n-1)$ *cubes.*

Proof. In dimension t, $\mathscr{M}(G., n)_A$ is given by the intersection

$$\mathscr{M}(G., n)_{A,t} = \bigcap \{\operatorname{Ker} d_{i+t}^{n+t} : i \in A\}.$$

The t dimensional part of the Moore complex of $\mathscr{M}(G., n)_A$ is thus

$$N(\mathscr{M}(G., n)_A)_t = \bigcap \{\operatorname{Ker} d_j^{n+t} : j = 1, \ldots, t\} \cap \bigcap \{\operatorname{Ker} d_{i+t}^{n+t} : i \in A\}$$

Writing $A(t) = \{1., , , , . t\} \cup \{i + t : i \in A\} \subseteq \langle n + t \rangle$, then if $A \neq \langle n \rangle, A(t) \neq \langle n + t \rangle$, the lemma of Proposition 8 applies and the t-dimensional homology vanishes if $t > 0$. The 0-dimensional homology of $N(\mathscr{M}(BG., n)_A)$ is also checked to be $M(G., n)_A$ as claimed.

If $A = \langle n \rangle$ then $A(t) = \langle n + t \rangle$ and $N(\mathscr{M}(G., n)_{\langle n \rangle})_t = NG_{n+t}$ as the differential in the two cases is the same. This proves (b).

For part (c), in dimension t, the claimed exact sequence is

$$1 \to \operatorname{Ker}_{1+t}^{n+1} \cap \bigcap \{\operatorname{Ker} d_{i+t}^{n+t} : i \in B(1\} \to \bigcap \{\operatorname{Ker} d_{i+t}^{n+t} : i \in B(1)\} \to$$
$$\bigcap \{\operatorname{Ker} d_{i+t}^{n-1+t} : i \in B\} \to 1$$

where $B \subseteq \langle n - 1 \rangle$ and $B(1) = \{b + 1 : b \in B\}$. The exactness of the statement is proved in a similar way to the proof of the lemma in Proposition 8. The details are omitted.

Remark. Using the intermediate décalage construction given by Illusie [11], a proof of (a) using the contractibility of the augmented décalage of a simplicial group may be given. Similarly a proof of (c) using the total décalage is quite easy to find.

PROPOSITION 11. *For any simplicial group, G., there is a natural isomorphism*

$$Q\mathscr{M}(G., n) \cong G.$$

where Q is the simplicial extension of the functor

$$Q : Inc.Crs^n \to Gps.$$

Proof. This follows from repeated application of (c) of Proposition 10 and the trivial fact that $\mathscr{M}(G., 0) = G.$,—by default.

4. THE DIAGONAL OF THE MULTINERVE

Given any crossed module (= crossed 1-cube), $\mu: M \to P$, the corresponding cat^1-group is $(M \rtimes P, s, t)$ where $s(m, p) = p$ and $t(m. p) = \mu(m)p$ (c.f. Loday [12]). This cat^1-group has an internal category structure within the category of groups given by:

(m, p) and (m', p') are composable if and only if $p' = \mu(m)p$ and then the composite is $(m'm, p)$.

The order of composition is illustrated by

$$p \xrightarrow{(m,p)} \mu(m)p \xrightarrow{(m',\mu(m)p)} \mu(mm')p.$$

The nerve of this category will have simplicial group structure as the category structure is internal to the category of groups (all structure maps are homomorphisms). If $\mathcal{M} = (M, P, \mu)$, then this simplicial group is denoted $E(\mathcal{M})$ and is given by:

$$E(\mathcal{M})_n \cong M \rtimes (M \rtimes (\ldots (M \rtimes P) \ldots)) \qquad n \text{ copies of } M$$

$$d_0(m_n, \ldots, m_1, p) = (m_n, \ldots, m_2, \mu(m_1)p)$$

$$d_i(m_n, \ldots, m_1, p) = (m_n, \ldots, m_{i+1}m_i, \ldots, m_1, p) \qquad \text{if } 0 < i < n$$

$$d_n(m_n, \ldots, m_1, p) = (m_{n-1}, \ldots, m_1, p)$$

$$s_j(m_{n-1}, \ldots, m_1, p) = (m_{n-1}, \ldots, 1, \ldots, m_1, p)$$

where $0 \leq j \leq n-1$ and the identity element of $\mathcal{M}$ is inserted in the $(j+1)$st position.

If $\mathcal{M}$ is a crossed n-cube, there is an associated cat^n-group (Ellis and Steiner, [9]) and hence on applying E in the n-independent directions, this construction leads naturally to an n-simplicial group. Taking the diagonal of this "multinerve" give a simplicial group which will be denoted $E^{(n)}\mathcal{M}$.

In [12], Loday uses an analogous construction, but takes the nerve of the group structure as well, then takes the geometric realization of the diagonal of the resulting $(n+1)$-simplicial set. This is the classifying space of $\mathcal{M}$.

For $n = 1$, the functor $\mathcal{M}(-, 1)$ takes a simplicial group $G.$ to the crossed module

$$\frac{\operatorname{Ker} d_1}{d_0 NG_2} \to G_0$$

and E of this crossed module is $t_{1]}G$. Thus E in this dimension gives a way to recover $t_{1]}G$ and to associate to any $\mathcal{M}$ a simplicial group. In fact if $\mathcal{M}$ is a crossed module then $M(E(\mathcal{M}), 1)$ is isomorphic to $\mathcal{M}$ itself. This E provides a quasi-inverse for $M(-, 1)$ when restricted to $T_{1]}$.

As not all crossed n-cubes are isomorphic to ones of the form $M(G., n)$ if $n > 1$, and although $M(-, n)$ restricted to $T_{n]}$ is faithful, it seems difficult to verify that this functor is full i.e. the dimension 1 result does not generalize directly. Later the simplicial group version of Loday's n-type theorem will be given and this together with some of the subsidiary results that go to prove it, will provide a suitable generalization.

The following observations (due to Loday) provide the first step in this process:

PROPOSITION 12. *Given a crossed module $\mathcal{M} = (M, P, \mu)$, the simplicial group $E(\mathcal{M})$ has the following properties*:

(a) *if $M = 1$, $E(\mathcal{M})$ is the simplicial group $K(P, 0)$, having P in all dimensions and with all face and degeneracy maps the identity morphism on P*;

(b) *if $P = 1$, $E(\mathcal{M})$ is a $K(M, 1)$ i.e. $\pi_1(E(\mathcal{M})) = M$, $\pi_i(E(\mathcal{M})) = 0$ if $i \neq 1$*;

(c) *if μ is a monomorphism, there is a projection*

$$E(\mathcal{M}) \to K(P/M, 0)$$

whose fibre, $E(M, M, =)$, is naturally contractible;

(d) *in general, $E(\mathcal{M})$ fits in a natural fibration sequence (= short exact sequence)*

$$E(\mathcal{M}_0) \to E(\mathcal{M}) \to E(\mathcal{M}_2)$$

where $\mathcal{M}_0 = (\mathrm{Ker}\,\mu \to 1)$, $\mathcal{M}_2 = (\mathrm{Im}\,\mu \to P)$ and so are handled by previous cases.

Proof. The first thing to note is that as $NE(\mathcal{M})$ is a complex with

$$\begin{aligned} NE(\mathcal{M})_i &= 0 \qquad i > 1 \\ NE(\mathcal{M})_1 &= M \\ NE(\mathcal{M})_0 &= P \end{aligned}$$

and with $\partial = \mu\colon NE(\mathcal{M})_1 \to NE(\mathcal{M})_0$, the homotopy groups of $E(\mathcal{M})$ for an arbitrary $\mathcal{M}$ are easily calculated. In particular (b) follows immediately. Observation (a) is also immediate from the construction of E.

The importance of (c) and (d) is thus not for calculating the homotopy groups, but for the naturality statements. The following lemma will be needed in the general case as well.

Lemma B. *If*

$$1 \longrightarrow \mathcal{M}_0 \xrightarrow{\alpha} \mathcal{M}_1 \xrightarrow{\beta} \mathcal{M}_2 \longrightarrow 1$$

is a short exact sequence of crossed n-cubes (i.e. for each $A \subseteq \langle n \rangle$

$$1 \longrightarrow \mathcal{M}_{0,A} \xrightarrow{\alpha_A} \mathcal{M}_{1,A} \xrightarrow{\beta_A} \mathcal{M}_{2,A} \longrightarrow 1$$

is a short exact sequence of groups with α_A a normal monomorphism), then

$$1 \longrightarrow E^{(n)}(\mathcal{M}_0) \xrightarrow{E^n(\alpha)} E^{(n)}(\mathcal{M}_1) \xrightarrow{E^n(\beta)} E^{(n)}(\mathcal{M}_2) \longrightarrow 1$$

is an exact sequence of simplicial groups.

Proof of Lemma. For $n = 1$, it is by observation and hence for general n, the multinerve gives an exact sequence of n-simplicial groups. Applying the diagonal clearly gives the result.

Return to the proof of Proposition 11.

For (c) apply the lemma to the short exact sequence

$$\begin{array}{ccccc} M & \longrightarrow & M & \longrightarrow & 1 \\ {\scriptstyle =}\downarrow & & {\scriptstyle \mu}\downarrow & & \downarrow \\ M & \xrightarrow[\mu]{} & P & \longrightarrow & P/M \end{array}$$

and then use (a) to determine $E(1 \to P/M)$.

This leaves verification of the statement about $E(M = M)$. This simplicial group has an "extra degeneracy" given by

$$s_{-1}(m_{n-1}, \ldots, m_1, m_0) = (m_{n-1}, \ldots, m_1, m_0, 1)$$

which gives a natural contraction. Thus $E(\mathcal{M}) \to K(P/M, 0)$ is a trivial fibration.

Proposition 13. *Given any simplicial group G and any $n \geq 0$, there is a natural epimorphism*

$$EM(G., n) \to t_{n]}G.$$

with naturally contractible kernel.

Proof. Without loss of generality, it will be assumed that G is in $T_{n]}$, the full subcategory of Simp.Gps. determined by those simplicial groups with $NG_1 = 0$ for $i > n$. For $n = 0$, the result is trivial as $EM(G., 1) \cong \pi_0(G) \cong t_{0]}G$.

For $n = 1$, as remarked above, $EM(G., 1)$ and $t_{1]}G.$ are isomorphic. As an inductive hypothesis, take the existence of such an epimorphism for $n - 1$, together with the provision that this be compatible with crossed module structures i.e. preserves action, etc. This is trivial for the dimensions already examined.

Given $G.$ and n, the crossed n-cube $M(G., n)$ can be viewed as a crossed module of crossed $(n - 1)$-cubes

$$M(K., n-1) \xrightarrow{\mu_n} M(\mathrm{Dec}\, G., n-1)$$

where $K. = \mathrm{Ker}(d_{\mathrm{last}}\colon \mathrm{Dec}\, G. \to G.)$. The inductive hypothesis thus gives a diagram

$$\begin{array}{ccc}
E^{(n-1)}M(K., n-1) & \longrightarrow & t_{n-1]}K. \\
\downarrow & & \downarrow \\
E^{(n-1)}M(\mathrm{Dec}G., n-1) & \longrightarrow & t_{n-1]}\mathrm{Dec}\, G.
\end{array}$$

in which the vertical maps are simplicial crossed modules and the horizontal maps, epimorphisms with naturally contractible kernels, by which is implied that the contractions are compatible with the vertical map.

Taking the nerve in the vertical, i.e. the crossed module, direction gives a morphism of bisimplicial groups. Applying the diagonal on the left gives $E^{(n)}M(G., n)$. On the right the bisimplicial group is as follows (illustrated in low dimensions)

$$\begin{array}{ccccccc}
 & \downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow \\
\begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} & K_2 \rtimes K_2 \rtimes G_3 & \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} & K_2 \rtimes G_3 & \rightrightarrows & G_3 \\
 & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} & K_1 \rtimes K_1 \rtimes G_2 & \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} & K_1 \rtimes G_2 & \rightrightarrows & G_2 \\
 & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} & K_0 \rtimes K_0 \rtimes G_1 & \rightrightarrows & K_0 \rtimes G_1 & \longrightarrow & G_1
\end{array}$$

The decomposition $G_n \cong K_{n-1} \rtimes G_{n-1}$ may be substituted into this and on taking the diagonal of the result, one obtains

$$\cdots \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} K_2 \rtimes K_2 \rtimes K_2 \rtimes G_2 \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} K_1 \rtimes K_1 \rtimes G_1 \rightrightarrows K_0 \rtimes G_0$$

Explicit descriptions of all the d_i and s_j can be given as well as a split epimorphism from this simplicial group to $t_{n]}G$. The kernel of this epimorphism is the simplicial group that, in dimension n, consists of all $(k_n, \ldots, k_0, 1)$ with all $k_i \in k_n$. The natural contraction, viewed as usual as an "extra degeneracy" is given by

$$s_{-1}(k_n, \ldots, k_0, 1) = (s_0 k_n, \ldots, s_0 k_0, 1, 1).$$

Combining this epimorphism with the one induced from the inductive hypothesis completes the inductive step, except to note that it is clear that it respects the crossed module structures necessary to apply it at the next level if necessary.

Note that the above proposition shows how near $M(-, n)$ is to being full and hence an embedding. Given $G.$ and $H.$ and a crossed n-cube morphism $f\colon M(G., n) \to M(H., n), f$ induces by the above process a simplicial group morphism $\tilde{f}\colon t_{n]}G. \to t_{n]}H.$, but it is not clear that f and $M(\tilde{f}, n)$ are the same or by how much they can differ.

A similar result holds with M replacing $\mathcal{M}$. Recall that

$$\mathcal{M}_n = \mathcal{M}(-, n)\colon \text{Simp.Gps.} \to \text{Simp.(Inc.Crs}^n)$$

and

$$\mathcal{Q}\colon \text{Simp.(Inc.Crs}^n) \to \text{Simp.Gps.}$$

satisfy

$$\mathcal{Q}\mathcal{M}_n \cong \text{Id by Proposition 11.}$$

Applying $E^{(n)}\colon \text{Crs}^n \to \text{Simp.Gps.}$ dimension wise to a simplicial inclusion crossed n-cube yields a bisimplicial group and on using the diagonal thus gives a simplicial group. The resulting composite functor will be denoted $D^{(n)}$.

The following proposition compares $D^{(n)}$ with $\mathcal{Q}$.

Proposition 14. *There is a natural epimorphism*

$$D^{(n)} \to \mathcal{Q}$$

such that for any simplicial inclusion crossed n-cube $\mathcal{M}$,

$$D^{(n)}\mathcal{M} \to \mathcal{Q}\mathcal{M}$$

has a (naturally) contractible kernel. In particular $D^{(n)}\mathcal{M}(G., n)$ is homotopically equivalent to $G.$ for any simplicial group, $G.$.

Proof. For $n = 0$, there is nothing to prove.

For $n = 1$, by Proposition 12(c), if $M \lhd P$ is considered as an inclusion crossed module, $\mathcal{M}$, then

$$E(\mathcal{M}) \xrightarrow{\varphi} E(1 \to \mathcal{Q}(\mathcal{M}))$$

is an epimorphism with contractible kernel. Proposition 12(a) states that $E(1 \to \mathcal{Q}(\mathcal{M})) = K(\mathcal{Q}(M), O)$, so applying this levelwise to a simplicial inclusion crossed module yields the required epimorphism. The natural contractibility of the kernel follows from the natural contractibility in one direction of the bisimplicial subgroup that is the kernel of the levelwise morphism, $\varphi.$. The result for $n = 1$ follows.

Now assume the result for $n - 1$ and let $\mathcal{M}$ be a simplicial inclusion crossed n-cube. Consider as before the nth direction crossed module of crossed $(n-1)$-cubes:

$$\mu_n\colon \mathcal{M}_1 \to \mathcal{M}_0$$

Using the exact sequence

$$\begin{array}{ccccc} \mathcal{M}_1 & \xrightarrow{=} & \mathcal{M}_1 & \longrightarrow & 1 \\ = \downarrow & & \downarrow \mu_n & & \downarrow \\ \mathcal{M}_1 & \xrightarrow{\mu_n} & \mathcal{M}_0 & \longrightarrow & \operatorname{Coker} \mu_n \end{array}$$

and the exactness of $E^{(n)}$, (Lemma B),

$$D^{(n)}(\mathcal{M}) \to D^{(n)}(1 \to \operatorname{Coker} \mu_n)$$

is an epimorphism with naturally contractible kernel, namely $D^{(n)}(\mathcal{M}_1 \xrightarrow{=} \mathcal{M}_1)$. However $D^{(n)}(1 \to \operatorname{Coker} \mu_n) \cong D^{(n-1)}(\operatorname{Coker} \mu_n)$ and $\operatorname{Coker} \mu_n$ is a simplicial inclusion crossed

$(n-1)$-cube that has a quotient $\mathscr{Q}(\mathscr{M})$. By hypothesis the natural map from $D^{(n-1)}(\operatorname{Coker}\mu_n)$ is an epimorphism with naturally contractible kernel, hence so also is the composite from $D^{(n)}(\mathscr{M})$ to $\mathscr{Q}(\mathscr{M})$. As $\mathscr{Q}\mathscr{M}.(G., n) \cong G.$, there is nothing left to prove.

5. THE FUNCTOR, $\mathscr{C}$.

Proposition 6 showed that applying π_0 to a simplicial crossed n-cube gave a crossed n-cube. The aim of this section is to prove that up to isomorphism all crossed n-cubes arise in this way. The proof will depend on a construction given by Loday in [12].

Suppose $\mathscr{M} = (M, P, \mu)$ is a crossed module, then there is a short exact sequence

$$\begin{array}{ccccc} 1 & \longrightarrow & M & \longrightarrow & M \\ \downarrow & & \partial\downarrow & & \downarrow\mu \\ M & \xrightarrow{\varepsilon} & M\rtimes P & \xrightarrow{t} & P \end{array}$$

of crossed modules, where $\varepsilon(m) = (m^{-1}, \mu(m))$, $\partial(m) = (m, 1)$ and $t(m, p) = \mu(m)p$, so it is the target map of the cat^1-group structure on $M\rtimes P$. Applying E to each term gives a short exact sequence of simplicial groups. Writing $\Gamma_1(\mathscr{M}) = (1, M, 1)$ and $\Gamma_0(\mathscr{M}) = (M, M\rtimes P, \partial)$ the corresponding exact sequence is

$$E\Gamma_1\mathscr{M} \to E\Gamma_0\mathscr{M} \to E\mathscr{M}$$

The calculations made in Proposition 12 show

(i) $\qquad \pi_0(E\Gamma_1\mathscr{M}) \cong M, \quad \text{and} \quad \pi_i(E\Gamma_1\mathscr{M}) = 0 \quad \text{if } i > 0$

(ii) $\qquad \pi_0(E\Gamma_0\mathscr{M}) \cong P, \quad \text{and} \quad \pi_i(E\Gamma_0\mathscr{M}) = 0 \quad \text{if } i > 0$

whilst a direct calculation shows that $\varepsilon*$ induces μ up to these two isomorphisms. This proves the case $n = 1$ of the following theorem:

THEOREM 1. *Given any crossed n-cube, $\mathscr{M}$, there is a simplicial inclusion crossed n-cube $\mathscr{E}(\mathscr{M})$ satisfying*

(i) *for each $A \subseteq \langle n\rangle$, $\pi_0(\mathscr{E}(\mathscr{M})_A) \cong \mathscr{M}_A$ and $\pi_1(\mathscr{E}(\mathscr{M})_A) \cong 0$ if $i > 0$*

(ii) $\pi_0\mathscr{E}(\mathscr{M}) \cong \mathscr{M}$ *in* Crs^n.

The assignment of $\mathscr{E}(\mathscr{M})$ to $\mathscr{M}$ is functorial and the composite $\pi_0\mathscr{E}$ is naturally isomorphic to the identity on Crs^n.

Proof. Noting that the above operations Γ_1 and Γ_0 are functors on crossed modules and that ε: $\Gamma_1 \to \Gamma_0$ is a natural transformation, define functors Γ_m^i on Crs^n for $i \in \langle n\rangle$, $m = 0$ or 1 by applying Γ_m in direction i. If $i \neq j$ and for any $\ell, m \in \{0, 1\}$, $\Gamma_\ell^i\Gamma_m^j = \Gamma_m^j\Gamma_\ell^i$.

If $\mathscr{M}$ is a crossed n-cube and $A \subseteq \langle n\rangle$, there is a crossed n-cube $\Gamma_A\mathscr{M}$ defined by

$$\Gamma_A\mathscr{M} = \Gamma_{\alpha(n)}^n \ldots \Gamma_{\alpha(1)}\mathscr{M}$$

where

$$\Gamma_{\alpha(i)}^i = \begin{cases} \Gamma_1^i & \text{if } i \in A \\ \Gamma_0^i & \text{if } i \notin A \end{cases}$$

If $i \in A$, define v_i: $\Gamma_A\mathscr{M} \to \Gamma_{A-\{i\}}\mathscr{M}$ by using ε in the ith direction. Explicitly

$$\mu_i = \Gamma_{\alpha(n)}^n \ldots \Gamma_{\alpha(i+1)}^{i+1}\varepsilon^i\Gamma_{\alpha(i-1)}^{i-1} \ldots \Gamma_{\alpha(1)}^1$$

where ε^i: $\Gamma_1^i \to \Gamma_0^i$ sends m to $(m^{-1}, \mu_i(m))$ if $i \in A$ and is the identity otherwise.

Taking commutators as the h-maps gives $\{\Gamma_A \mathcal{M}\colon A \subseteq \langle n\rangle, \{v_i\}, h\}$ the structure of a crossed n-cube of crossed n-cubes (thus an object in Crs^{2n}). Now let $\mathscr{E}(\mathcal{M}) = \{(E^{(n)}\Gamma_A \mathcal{M})\colon A \subseteq \langle n\rangle\}$ where the functor $E^{(n)}$ is applied in the old n-directions. This $\mathscr{E}(\mathcal{M})$ is a simplicial inclusion crossed n-cube.

This $\mathscr{E}$ is a functor. A simple inductive proof then gives $\pi_0 \mathscr{E}(\mathcal{M}) \cong \mathcal{M}$ and the other points about the higher homotopy groups of the $\mathscr{E}(\mathcal{M})_A$.

The relationship between $E, \mathscr{E}$ and Q is simple to give.

PROPOSITION 15. *There is a natural isomorphism $\mathscr{Q}\mathscr{E} \cong E$.*

Proof. At the start of this section it was noted that

$$E\Gamma_1 \mathcal{M} \to E\Gamma_0 \mathcal{M} \to E\mathcal{M}$$

was a short exact sequence; this implies $\mathscr{Q}E\Gamma\mathcal{M} \cong E\mathcal{M}$. Applying this argument in each direction of a crossed n-cube gives the required general result.

The final result on the relationship of $\mathscr{E}$ with other functors is the following which is the algebraic analogue of Loday's Theorem 1.4 [12].

PROPOSITION 16. *There is a functor*

$$\mathscr{H}\colon Simp(Inc.Crs^n) \to Simp(Inc.Crs^n)$$

with natural transformations

$$\delta\colon \mathscr{H} \to \mathrm{id}$$

$$\delta'\colon \mathscr{H} \to \mathscr{E}\pi_0$$

such that for each $A \subseteq \langle n\rangle$ and each simplicial inclusion crossed n-cube $\mathcal{M}$, $\delta(M)_A$ and $\delta'(\mathcal{M})_A$ induce isomorphisms on π_0. In particular on the subcategory $t_{0]}$ Simp(Inc.Crs^n) in which for each $\mathcal{M}$ and $A \subseteq \langle n\rangle$, M_A is homotopic to a $K(\pi, 0)$ for some group, π, the natural maps $\delta(\mathcal{M})_A$ and $\delta'(\mathcal{M})_A$ are natural homotopy equivalences.

Proof. Let $\mathcal{M}$ be in Simp(Inc.Crs^n). Apply $E^{(n)}\Gamma$ to each level to get a bisimplicial inclusion crossed n-cube. Taking π_0 in one direction gives $\mathcal{M}$, whilst in the other it gives $E^{(n)}\Gamma\pi_0(\mathcal{M}) = \mathscr{E}\pi_0(\mathcal{M})$. Setting $\mathscr{H} = \operatorname{diag} E^{(n)}\Gamma(\mathcal{M})$ and δ, and δ' the induced maps, provides the structure for the theorem. The details are then routine to check.

COROLLARY. *If $G.$ is any simplicial group within $T_{n]}$, then there are natural homotopy equivalences*

$$\mathscr{Q}\mathscr{H}\mathcal{M}(G., n) \to G.$$

and

$$\mathscr{Q}\mathscr{H}\mathcal{M}(G., n) \to EM(G., n)$$

Proof. Applying Proposition 16 to the simplicial inclusion crossed n-cube $\mathcal{M}(G., n)$, which, as $G.$ is in $T_{n]}$, is itself in $t_{0]}$ Simp(Inc.Crs^n) gives

$$\mathcal{M}(G., n) \xleftarrow{\simeq} \mathscr{H}\mathcal{M}(G., n) \xrightarrow{\simeq} \mathscr{E}\pi_0\mathcal{M}(G., n)$$

Now apply the functor $\mathscr{Q}$ and use Propositions 11 and 15 to get

$$G. \xleftarrow{\simeq} \mathscr{Q}\mathscr{H}\mathcal{M}(G., n) \xrightarrow{\simeq} EM(G., n)$$

The only remaining fact to check is that $\mathscr{D}$ respects homotopy equivalences and this is routine.

6. QUASI-ISOMORPHISMS OF CROSSED n-CUBES

A morphism $f\colon \mathscr{M} \to \mathscr{N}$ of crossed n-cubes will be called an epimorphism if for each $A \subseteq \langle n \rangle, f_A$ is an epimorphism of groups. Such a morphism, as has been used implicitly several times already, induces an epimorphism, $E^{(n)} f$, of simplicial groups. As the notion of "epimorphism" and "fibration" are equivalent for simplicial groups, the morphism f will be called a trivial epimorphism if $E^{(n)} f$ is a trivial fibration i.e. is both a fibration and a homotopy equivalence. More precisely the term "trivial epimorphism" will be used for such a morphism f if $\operatorname{Ker} E^{(n)} f$ is contractible.

Suppose that $\mathscr{M}, \mathscr{N}$ are two functors from some category $\mathbf{C}$ to Crs^n and f_X: $\mathscr{M}(X) \to \mathscr{N}(X)$ is a natural transformation, then f will be said to be a natural trivial epimorphism if for each object X in $C, f_X\colon \mathscr{M}(X) \to \mathscr{N}(X)$ is a trivial epimorphism *and* the contractions on the kernels, $\operatorname{Ker} E^{(n)} f_X$, can be chosen to vary naturally with respect to X.

Let Σ denote the class of trivial epimorphisms in Crs^n and form $\mathrm{Crs}^n(\Sigma^{-1})$, the category of fractions with respect to Σ, which will also be denoted by $Ho(\mathrm{Crs}^n)$. A map f in Crs^n will be called a quasi-isomorphism if the corresponding map $[f]$ in $Ho(\mathrm{Crs}^n)$ is an isomorphism.

This has an analogue in $\mathrm{Simp}(\mathrm{Inc.Crs}^n)$. If f is an epimorphism in $\mathrm{Simp}(\mathrm{Inc.Crs}^n)$, then f will be said to be a trivial fibration if $\operatorname{Ker} f$ is contractible (this should be interpreted as meaning $D^n \operatorname{Ker} f$, that is $\operatorname{diag} sE^{(n)} \operatorname{Ker} f$, is contractible). Forming $Ho(\mathrm{Simp}(\mathrm{Inc.Crs}^n))$ by formally inverting the trivial fibrations, the term "quasi-isomorphism" may be extended for use in $\mathrm{Simp}(\mathrm{Inc.Crs}^n)$.

The main part of Section 7 of this paper will be devoted to proving the following two theorems.

THEOREM 2 (c.f. Loday [12]). *The functor $M(-, n)$: Simp.Gps. $\to$ Crs^n induces an equivalence of categories*

$$Ho_n(Simp.Gps). \xrightarrow{\simeq} Ho(Crs^n)$$

THEOREM 3. *The functor $\mathscr{M}(-, n)$: Simp.Gps. $\to$ (Simp.Inc.Crs^n) induces an equivalence of categories,*

$$Ho(Simp.Gps) \xrightarrow{\simeq} Ho(Simp.Inc.Crs^n)$$

Much of the proof of these two results has already been given. The situation can be summarized in the following diagram:

$$\begin{tikzcd}
& \mathrm{Simp}(\mathrm{Inc.Crs}^n) \arrow[dl, shift left, "\mathscr{D}"] \arrow[dr, shift left, "\pi_0"] & \\
\mathrm{Simp.Gps} \arrow[ur, shift left, "\mathscr{M}"] \arrow[rr, shift left, "M"] & & \mathrm{Crs}^n \arrow[ul, shift left, "\mathscr{E}"] \arrow[ll, shift left, "E"]
\end{tikzcd}$$

together with the information:

— $T_{n]} \subset$ Simp.Gps. is a reflexive subcategory with reflection $t_{n]}$ *and* $t_{n]}$ is an n-equivalence (Proposition 3).

— $M = \pi_0 \mathcal{M}$ by definition
— $\mathcal{Q}\mathcal{M} \cong \mathrm{Id}$ (Proposition 11)
— There is a natural trivial fibration

$$EM \to t_{n]}$$

— There is a functor $D^{(n)}$: Simp(Inc.Crs^n) $\to$ Simp.Gps. so that $D^{(n)}$ is quasi-isomorphic to $\mathcal{Q}$ (Proposition 14)
— $\pi_0 \mathcal{E} \cong \mathrm{Id}$ (Theorem 1)
— $\mathcal{Q}\mathcal{E} \cong E$ (Proposition 15)
— There is a functor $\mathcal{H}$: Simp(Inc.Crs^n) $\to$ Simp(Inc.Crs^n)

with natural transformations

$$\delta\colon \mathcal{H} \to \mathrm{Id}$$
$$\delta'\colon \mathcal{H} \to \mathcal{E}\pi_0$$

so that δ and δ' induce isomorphisms on π_0 (Proposition 16), thus $\mathcal{Q}\mathcal{H}\mathcal{M}(G, n) \simeq G$ and $\mathcal{Q}\mathcal{H}\mathcal{M}(G., n) \simeq EM(G_{\cdot}, n)$ if $G \in T_{n]}$.

For Theorem 2, it therefore will be necessary to enquire into the composite ME whilst for Theorem 3, $\mathcal{M}\mathcal{Q}$ will need studying.

7. THE COMPOSITE $\mathcal{M}\mathcal{Q}$.

PROPOSITION 17. (Adapted from an idea of Conduché). *Let*

$$\begin{array}{ccc} M \cap N & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\nu} & P \end{array}$$

be an inclusion crossed square. Let N act on M via P and let $\Delta\colon M \cap N \to M \rtimes N$ *be the twisted diagonal* $\Delta(m) = (m^{-1}, m)$*. Then* $\operatorname{Im}\Delta$ *is a normal subgroup of* $M \rtimes N$ *and the morphisms* μ, ν *induce an inclusion crossed module*

$$\frac{M \rtimes N}{\operatorname{Im}\Delta} \xrightarrow{\partial} P$$

where

$$\partial((m, n)\operatorname{Im}\Delta) = \mu(m)\,\nu(n)$$

Proof. The map $\partial\colon M \rtimes N \to P$ given by $\partial(m, n) = \mu(m)\nu(n)$ is easily checked to be a group homomorphism. Its kernel is the subgroup consisting of those pairs (m, n) such that $\mu(m) = \nu(n)^{-1}$, but as μ and ν are inclusions, this is precisely $\operatorname{Im}\Delta$. Thus ∂ is a monomorphism. (In fact it is equivalent as a sub object of P to the inclusion of MN).

PROPOSITION 18. *Given P and normal subgroups M, N as above, there is a natural map of crossed modules*

$$\begin{pmatrix} MN \\ \partial \downarrow \\ P \end{pmatrix} \xrightarrow{q} \begin{pmatrix} N/M \cap N \\ \downarrow \bar{\nu} \\ P/M \end{pmatrix}$$

given by the evident quotient morphism. The simplicial group E Kerq is naturally contractible (so q is a trivial fibration in Simp(Inc.Crs)).

Proof. This is the Noether isomorphism theorem in disguise. The kernel is isomorphic to (M, M, id) and so E of it is clearly naturally contractible.

This proposition thus gives a zig-zag.

$$\begin{pmatrix} N/M\cap N \\ \downarrow \bar{\nu} \\ P/M \end{pmatrix} \xleftarrow{q_1} \begin{pmatrix} MN \\ \downarrow \\ P \end{pmatrix} \xrightarrow{q_2} \begin{pmatrix} M/M\cap N \\ \downarrow \bar{\mu} \\ P/N \end{pmatrix}$$

joining the horizontal and vertical quotients of the inclusion crossed square (of Proposition 17) via the "diagonal" crossed module given by that result. Thus the horizontal and vertical crossed modules are isomorphic within $Ho(\mathrm{Simp.(Inc.Crs^1)})$. The next step is to extend this simplicially to simplicial inclusion crossed squares.

PROPOSITION 19. *Given an exact sequence*

$$1 \longrightarrow M. \xrightarrow{\partial} P. \longrightarrow Q. \longrightarrow 1$$

of simplicial groups, then $(M., P., \partial)$ *and* $\mathcal{M}(\mathcal{Q}., 1)$ *are naturally isomorphic in* $Ho(Simp.(Inc.Crs^1))$.

Proof. Interpreting the monomorphism ∂ as a simplicial inclusion crossed module, apply $\mathcal{M}(-, 1)$ to $M.$, $P.$ and $Q.$

This gives a diagram of simplicial groups

$$\begin{array}{ccccc}
\mathrm{Ker}\,\delta_0^M & \longrightarrow & \mathrm{Ker}\,\delta_0^P & \longrightarrow & \mathrm{Ker}\,\delta_0^Q \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Dec}^1 M. & \longrightarrow & \mathrm{Dec}^1 P. & \longrightarrow & \mathrm{Dec}^1 Q. \\
\downarrow & & \downarrow & & \downarrow \\
M. & \longrightarrow & P. & \longrightarrow & Q.
\end{array}$$

in which all rows and columns are exact. The top left hand square is a simplicial inclusion crossed square. Applying Propositions 17 and 18 gives a simplicial inclusion crossed module $(R., \mathrm{Dec}^1 P, \mathrm{inc})$ where $R. = (\mathrm{Dec}^1 M.)\,(\mathrm{Ker}\,\delta_0^P)$, and quotient maps

$$\begin{pmatrix} M. \\ \downarrow \\ P. \end{pmatrix} \xrightarrow{q_1} \begin{pmatrix} R. \\ \downarrow \\ \mathrm{Dec}^1 P. \end{pmatrix} \xrightarrow{q_2} \begin{pmatrix} \mathrm{Ker}\,\delta_0^Q \\ \downarrow \\ \mathrm{Dec}^1 Q. \end{pmatrix} = \mathcal{M}(\mathcal{Q}, 1)$$

whose kernels are "contractible" at each level.

The next result gives the natural extension of Proposition 19 to simplicial inclusion crossed n-cubes.

THEOREM 4. *Let* $\mathcal{M}$ *be a simplicial inclusion crossed n-cube, then there is a natural zig-zag.*

$$\mathcal{M} \leftarrow R_n \rightarrow \mathcal{M}_1\, \mathcal{Q}_n \mathcal{M} \leftarrow \dots \rightarrow R_1 \rightarrow \mathcal{M}_n \mathcal{Q}\mathcal{M}$$

of morphisms joining it to $\mathcal{M}(\mathcal{Q}(\mathcal{M}), n)$. *Each of these morphisms is a natural trivial fibration of simplicial inclusion crossed n-cubes (i.e. natural epimorphism with naturally contractible kernel).*

Proof. First some notational points, $\mathcal{M}_n(G.)$ has been used as shorthand for $\mathcal{M}(G., n)$ in the statement of the theorem. If $\mathcal{M}$ is a simplicial inclusion crossed n-cube, $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ will denote the effect of the 1-dimensional quotienting operation in direction $1, 2, \ldots, n$ respectively. (For instance if $n = 2, \mathcal{Q}_1(\mathcal{M}) = (M_{\{1\}}/M_{\langle 2\rangle}, M_{\Phi}/M_{\{2\}}, \bar{\mu}_2), \mathcal{Q}_2(\mathcal{M}) = (M_{\{2\}}/M_{\langle 2\rangle}, M_{\varphi}/M_{\{1\}}, \bar{\mu}_1)$ whilst $\mathcal{Q}_2\mathcal{Q}_1 = \mathcal{Q}$, the total quotient operation).

Applying Propositions 5 and 9 to $\mathcal{M}$ (considered as a crossed $(n-1)$ cube of simplicial crossed modules) in the nth direction gives a short exact sequence

$$\{M_A \longrightarrow M_{A-\{n\}} \xrightarrow{q_n} \mathcal{Q}_n(M_{A-\{n\}})\}$$

with $A \subseteq \langle n\rangle, n \in A$ and then a simplicial inclusion crossed n-cube

$$R_n = \{r_n\colon R_{n,A} \longrightarrow \mathrm{Dec}^1 M_{A-\{n\}}\} \text{ and maps}$$

$$\begin{pmatrix} M_A \\ \downarrow \\ M_{A-\{n\}} \end{pmatrix}_{n\in A\subseteq\langle n\rangle} \longleftarrow \begin{pmatrix} R_{n,A-\{n\}} \\ \downarrow \\ \mathrm{Dec}^1 M_{A-\{n\}} \end{pmatrix}_{n\in A\subseteq\langle n\rangle} \longrightarrow \begin{pmatrix} \mathrm{Ker}\,\delta_0 \\ \downarrow \\ \mathrm{Dec}^1 \mathcal{Q}_n(M)_{A-\{n\}} \end{pmatrix}_{n\in A\subseteq\langle n\rangle} = \mathcal{M}_1(\mathcal{Q}_n(\mathcal{M})$$

These maps have (naturally) contractible kernels i.e. the functor $E^{(n)}$ applied to the kernels δ gives (naturally) contractible simplicial groups. This process is now repeated in direction $(n-1)$ or $\mathcal{M}(\mathcal{Q}_n(\mathcal{M}), 1)$ giving

$$\mathcal{M}_1(\mathcal{Q}_n(\mathcal{M})) \longleftarrow R_{n-1} \longrightarrow \mathcal{M}_1(\mathcal{Q}_{n-1}(\mathcal{M}_1(\mathcal{Q}_n(\mathcal{M})))).$$

The right hand term here is isomorphic to $\mathcal{M}((\mathcal{Q}_{n-1}\mathcal{Q}_n(\mathcal{M})), 2) = \mathcal{M}_2(\mathcal{Q}_{n-1}\mathcal{Q}_n(\mathcal{M}))$ (this is the third isomorphism theorem for groups!).

Repeating in the other directions in turn gives the result on noting that

$$\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2 \ldots \mathcal{Q}_1.$$

Proof of Theorem 3. It has already been noted that $\mathcal{Q}\mathcal{M}_n = \mathrm{Id}$ (Proposition 11). Now Theorem 4 gives that there is a natural isomorphism between $\mathcal{M}_n\mathcal{Q}$ and Id within Ho(Simp. Inc. Crs^n).

Proof of Theorem 2. It remains to prove that there is a natural isomorphism between $M(E(M), n)$ and M within Ho(C) for $\mathcal{M}$ a crossed n-cube, since the natural maps $EM(G, n) \longrightarrow t_{n]} G \longleftarrow G$ clearly give an isomorphism between EM and Id inHo_n(Simp.Gps.).

Using Theorem 4, there is a zig-zag of trivial fibrations joining $\mathcal{E}(\mathcal{M})$ and $\mathcal{M}(\mathcal{Q}\mathcal{E}(\mathcal{M}), n)$ within Simp(Inc.Crs^n). Applying π_0 to this zig-zag gives a similar zig-zag of trivial fibrations (within Crs^n this time,) joining $M \cong \pi_0\mathcal{E}(M)$ and $\pi_0\mathcal{M}(\mathcal{Q}\mathcal{E}(M, n)) \cong M(E(M), n)$ since $M(-, n) = \pi_0\mathcal{M}(-, n)$ by definition whilst $\mathcal{Q}\mathcal{E} = E$ (Proposition 15). This completes the proof.

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School of Mathematics
University College of North Wales
Dean Street
Bangor
Gwynedd LL57 1UT
N. Wales
U.K.